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$C^{1,\alpha}$-regularity for surfaces with $H \in L^p$

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Abstract

In this paper we prove several results on the geometry of surfaces immersed in $\mathbb{R}^3$ with small or bounded L^2 norm of $|A|$. For instance, we prove that if the L^2 norm of $|A|$ and the L^p norm of H, $p > 2$, are sufficiently small, then such a surface is graphical away from its boundary. We also prove that given an embedded disk with bounded L^2 norm of $|A|$, not necessarily small, then such a disk is graphical away from its boundary, provided that the L^p norm of H is sufficiently small, $p > 2$. These results are related to previous work of Schoen-Simon [12] and Colding-Minicozzi [4].

1 Introduction

Inspired by the ideas of Schoen-Simon in [12] and Colding-Minicozzi in [4], in this paper we prove several results on the geometry of surfaces with small or bounded L^2 norm of $|A|$, where $|A| = \sqrt{k_1^2 + k_2^2}$ denotes the norm of the second fundamental form; k_1, k_2 are the principal curvatures.

Throughout this paper, M will be a smooth, compact, oriented surface with boundary, immersed in $\mathbb{R}^3$. Given $x \in M$, we let $\mathcal{B}_R(x)$ and $B_R(x)$ denote the intrinsic and extrinsic balls of radius R centered at x. Often and when the center of these balls is clear from the context we will write $\mathcal{B}_R$ and B_R instead of $\mathcal{B}_R(x)$ and $B_R(x)$. We let $H = k_1 + k_2$ denote the mean curvature.

One of the main theorems of this paper, Theorem 1.1 below, states that if $\mathcal{B}_R$ is an embedded disk with bounded L^2 norm of $|A|$, then $\mathcal{B}_R$ is graphical away from its boundary, provided that the L^p norm of H is sufficiently small, $p > 2$. This is related to previous results by Colding-Minicozzi for minimal surfaces [4], see also [2]. These previous results assume stronger conditions on H and deliver point-wise estimates for $|A|$. Clearly, one cannot expect point-wise estimates for $|A|$ with our assumptions.

Let $\nu\colon M \to S^2$ denote the Gauss map and let

$$g(x) = \sqrt{1 - \nu(x) \cdot e_3} = \frac{1}{\sqrt{2}}|\nu(x) - e_3|, \quad x \in M.$$

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Theorem 1.1. *Given $K > 0$ and $p > 2$, there exist $\varepsilon = \varepsilon(K,p)$ and $\gamma = \gamma(K)$, such that the following holds. Let $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$ be an embedded disk such that*

$$\int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2 \leq K r^4 \quad \text{and} \quad R^{\frac{p-2}{p}} \left(\int_{\mathcal{B}_R} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq \varepsilon r^2$$

for some $r \in [0, 1/4]$. Then, after a rotation,

$$\sup_{\mathcal{B}_{\gamma R}} g \leq \frac{5}{4} r.$$

Remark 1.2. *In* Section 4 *we actually prove a slightly more general version of* Theorem 1.1 *that does not require $\mathcal{B}_R$ to be a disk, cf.* Theorem 4.4.

A key ingredient in proving Theorem 1.1 and indeed an interesting geometric result in its own right, is Theorem 1.3 below; it states that if the L^2 norm of $|A|$ and the L^p norm of H are sufficiently small, $p > 2$, then a geodesic ball is graphical away from its boundary. In contrast to Theorem 1.1, in this result we are neither assuming that the geodesic ball is a disk nor that it is embedded. This theorem was motivated by a classical result of Schoen-Simon for surfaces with quasi-conformal Gauss map [12]. It is also related to the Choi-Schoen Curvature Estimate for minimal surfaces [3] and our extension of it to surfaces with "small" mean curvature [2].

Theorem 1.3. *There exist constants $c_1 > 0$ and $\beta \in (0, \frac{1}{2})$ such that the following holds. Given $p > 2$ there exists $c_2 = c_2(p)$ such that if $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$ is such that*

$$\int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2 \leq c_1 r^2 \quad \text{and} \quad \|H\|_{L^p(\mathcal{B}_R)} R^{\frac{p-2}{p}} \leq c_2 r$$

for some $r \in [0, 1]$ then, after a rotation,

$$\sup_{\mathcal{B}_{\beta R}} g \leq r.$$

Note that the assumption on the mean curvature is necessary as the C^1 norm of a smooth function over a bounded domain in $\mathbb{R}^2$ is not in general bounded by its $W^{2,2}$ norm.

In Corollary 3.2 we prove a similar result for surfaces with bounded, not necessarily small, L^p norm of H. In this case the L^2 bound for $|A|$ depends on the L^p bound for H.

Using Theorem 1.3, we can immediately prove an analogous result with the intrinsic ball replaced by an extrinsic one.

Corollary 1.4. *Let M be an orientable surface containing the origin with $\partial M \subset \partial B_R(0)$,*

$$\int_M |A|^2\, d\mathcal{H}^2 \leq c_1 r^2 \quad \text{and} \quad \|H\|_{L^p(M\cap B_R)} R^{\frac{p-2}{p}} \leq c_2 r$$

for some $r \in [0, \frac{1}{\sqrt{3}}]$, $p > 2$. Then, if M_R is a connected component of $M \cap B_{\frac{\beta R}{2}}$ containing the origin, after a rotation

$$\sup_{M_R} g \leq r,$$

where the constants c_1, c_2 and β are the constants in Theorem 1.3*.*

Our main theorems deal with intrinsic balls and, as an immediate consequence, we obtain results such as Corollary 1.4, where stronger hypotheses on the extrinsic geometry of M are assumed. Among many other crucial findings, several results related to Corollary 1.4 can be found in [1, 12, 16].

After having proved our main results, the $C^{1,\alpha}$-regularity follows from standard PDE theory, see [7] (cf. Remark 5.3).

2 Some results on the topology of $\mathcal{B}_R$

In order to prove the main theorems, we first need to prove some more general results on the geometry and topology of a geodesic ball with small (or bounded) L^2 norm of the second fundamental form. For this we recall the isoperimetric inequality (see Poincare inequality, [15, Theorem 18.6] and [9]):
For any open subset F of M, with $\overline{F} \subset M \setminus \partial M$, it is

$$|F|^{1/2} \leq C \left(|\partial F| + \int_F |H|\, d\mathcal{H}^2 \right), \tag{1}$$

where C is an absolute constant and $|F|$, $|\partial F|$ denote the area of F and the length of ∂F respectively.

In the rest of the paper we will denote by $|U|$ the n-dimensional Hausdorff measure of U, whenever U is a set of Hausdorff dimension n, as we did above with $|F|$ and $|\partial F|$.

The first lemma is a lower bound for the area of a surface, whose mean curvature has bounded L^2 norm.

Lemma 2.1. *If $\mathcal{B}_\rho := \mathcal{B}_\rho(x_0) \subset M \setminus \partial M$ and $C\|H\|_{L^2(\mathcal{B}_\rho)} \leq \frac{1}{2}$, where C is the isoperimetric constant given in* (1)*, then*

$$|\mathcal{B}_\rho| \geq \frac{1}{16C^2}\rho^2.$$

Proof. The isoperimetric inequality (1), with $F = \mathcal{B}_\rho$ gives

$$\begin{aligned} |\mathcal{B}_\rho|^{\frac{1}{2}} &\leq C\left(|\partial \mathcal{B}_\rho| + \int_{\mathcal{B}_r} |H|\, d\mathcal{H}^2\right) \\ &\leq C\left(\frac{d}{d\rho}|\mathcal{B}_\rho| + |\mathcal{B}_\rho|^{\frac{1}{2}}\|H\|_{L^2(\mathcal{B}_\rho)}\right). \end{aligned}$$

Thus, if $C\|H\|_{L^2(\mathcal{B}_\rho)} \leq \frac{1}{2}$ we obtain that

$$|\mathcal{B}_\rho|^{\frac{1}{2}} \leq 2C\frac{d}{d\rho}|\mathcal{B}_\rho| \implies \frac{d}{d\rho}(|\mathcal{B}_\rho|^{\frac{1}{2}}) \geq \frac{1}{4C}. \tag{2}$$

Integrating the equation above finishes the proof of the lemma. □

Recall that

$$H^2 = (k_1 + k_2)^2 = k_1^2 + k_2^2 + 2k_1k_2 \leq 2(k_1^2 + k_2^2) = 2|A|^2.$$

Therefore, if we assume $C\|A\|_{L^2(\mathcal{B}_\rho)} \leq \frac{1}{4}$, then the hypothesis on H in Lemma 2.1 and thus the conclusion of the lemma still hold.

It follows from the work in [6, 8, 13, 14] that for almost all $s > 0$,

$$\frac{d}{ds}|\partial \mathcal{B}_s| \leq 2\pi\chi(\mathcal{B}(s)) - \int_{\mathcal{B}_s} K\, d\mathcal{H}^2 \tag{3}$$

where $\chi(\mathcal{B}(s))$ denotes the Euler characteristic of $\mathcal{B}(s)$ (see also [4, 10]). We are now going to use Lemma 2.1 and equation (3) to study the topology of geodesic balls with small total curvature.

Given $\mathcal{B}_R \subset M \setminus \partial M$ let

$$\begin{aligned} T_1 &= \{\rho \in [0, R] : \chi(\mathcal{B}_\rho) = 1\} \\ T_0 &= \{\rho \in [0, R] : \chi(\mathcal{B}_\rho) \leq 0\} \end{aligned}$$

and for $i = 0, 1$, define $\alpha_i \in [0, 1]$ to be such that $|T_i| = \alpha_i R$. Since

$$\chi(\mathcal{B}_\rho) = 2 - \kappa - 2\mathrm{g}, \tag{4}$$

where κ is the number of components of $\partial \mathcal{B}_\rho$ and g is the genus, we have that $\chi(\mathcal{B}_\rho) \leq 1$ for all $\rho \in [0, R]$ and thus $|T_1| + |T_0| = R$ giving $\alpha_1 + \alpha_0 = 1$.

Using Lemma 2.1 and equation (2) in its proof, i.e.

$$|\mathcal{B}_\rho|^{\frac{1}{2}} \leq 2C\frac{d}{d\rho}|\mathcal{B}_\rho| = 2C|\partial \mathcal{B}_\rho|$$

and integrating equation (3) from 0 to R, we obtain that

$$\frac{R}{4C} \leq |\mathcal{B}_R|^{\frac{1}{2}} \leq 2C|\partial\mathcal{B}_R| \leq 4C\pi \int_0^R \chi(\mathcal{B}(s))ds - 2C \int_0^R \int_{\mathcal{B}_s} K \, d\mathcal{H}^2,$$

provided that $\|H\|_{L^2(\mathcal{B}_R)} \leq \frac{1}{2C}$. Since the Gauss equation gives

$$-K = \frac{|A|^2 - |H|^2}{2} \leq \frac{|A|^2}{2},$$

we obtain

$$\frac{R}{4C} \leq 4C\pi\alpha_1 R + CR \int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2.$$

Therefore, if $\int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2 \leq \frac{1}{8C^2}$ (which also implies that $\|H\|_{L^2(\mathcal{B}_R)} \leq \frac{1}{2C}$) we get

$$\frac{1}{32\pi C^2} \leq \alpha_1,$$

namely

$$|T_1| \geq \frac{1}{32\pi C^2} R.$$

Note that, by (4), if $\rho \in T_1$, then $\mathcal{B}_\rho$ is homeomorphic to a disk. Thus, we have proven the following lemma on the topology of geodesic balls with small L^2 norm of $|A|$.

Lemma 2.2. *Let $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$ be such that $\int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2 \leq \frac{1}{8C^2}$, where C is the isoperimetric constant given in* (1)*, then*

$$|T_1| \geq \frac{1}{32\pi C^2} R,$$

where $T_1 = \{\rho \in [0, R] : \chi(\mathcal{B}_\rho) = 1\}$.

The next lemma is an estimate from above for the area of a geodesic ball and the length of its boundary in terms of the L^2 norm of $|A|$.

Lemma 2.3. *If $\mathcal{B}_\rho := \mathcal{B}_\rho(x_0) \subset M \setminus \partial M$ then*

$$|\mathcal{B}_\rho| \leq \pi\rho^2 + \frac{1}{2}\rho^2 \int_{\mathcal{B}_\rho} |A|^2 \, d\mathcal{H}^2$$

and

$$|\partial\mathcal{B}_\rho| \leq 2\pi\rho + \frac{1}{2}\rho \int_{\mathcal{B}_\rho} |A|^2 \, d\mathcal{H}^2.$$

Proof. For any $s_0 \in [0, \rho]$, integrating equation (3) from 0 to s_0 gives

$$|\partial \mathcal{B}_{s_0}| \leq 2\pi \int_0^{s_0} \chi(\mathcal{B}_s) ds - \int_0^{s_0} \int_{\mathcal{B}_s} K \, d\mathcal{H}^2 \, , \, \forall s_0 \in (0, \rho]. \tag{5}$$

Integrating once more from 0 to ρ we have

$$|\mathcal{B}_\rho| \leq 2\pi \int_0^{\rho} \int_0^{s_0} \chi(\mathcal{B}_s) ds ds_0 - \int_0^{\rho} \int_0^{s_0} \int_{\mathcal{B}_s} K \, d\mathcal{H}^2. \tag{6}$$

Since $-K \leq \frac{|A|^2}{2}$ and $\chi(\mathcal{B}_s) \leq 1$ for all s we obtain from (5) with $s_0 = \rho$

$$|\partial \mathcal{B}_\rho| \leq 2\pi\rho + \frac{1}{2}\rho \int_{\mathcal{B}_\rho} |A|^2 \, d\mathcal{H}^2$$

and from (6)

$$|\mathcal{B}_\rho| \leq 2\pi \int_0^{\rho} s_0 ds_0 + \frac{1}{2}\rho^2 \int_{\mathcal{B}_\rho} |A|^2 \, d\mathcal{H}^2 \leq \pi\rho^2 + \frac{1}{2}\rho^2 \int_{\mathcal{B}_\rho} |A|^2 \, d\mathcal{H}^2.$$

□

3 Small total curvature implies graphical

In this section we prove Theorem 1.3 and its corollaries. Namely, we prove that for any geodesic ball, if the L^2 norm of $|A|$ and the L^p norm of H, $p > 2$, are sufficiently small, then such ball is graphical away from its boundary.

We begin by proving a lemma stating that given a compact surface M whose boundary satisfies certain geometric conditions, if the L^2 norm of $|A|$ and the L^p norm of H are sufficiently small, $p > 2$, then M is (locally) a graph over a fixed plane.

Recall the definition in the introduction; let $\nu \colon M \to S^2$ denote the Gauss map and let

$$g(x) = \sqrt{1 - \nu(x) \cdot e_3} = \frac{1}{\sqrt{2}} |\nu(x) - e_3|, \quad x \in M.$$

Note that if $g \leq \frac{1}{4}$ implies that M is locally graphical over the plane $\{x_3 = 0\}$ with gradient bounds (cf. Lemma 5.2). The core of the proof of the lemma follows the ideas in [12].

Lemma 3.1. *Given $p > 2$ there exists a constant $c_3 > 0$, depending only on p, such that the following holds. Let M be a compact orientable surface with boundary such that*

$$\int_M |A|^2 \, d\mathcal{H}^2 \leq \frac{\pi}{2} r^2 \quad \text{and} \quad \|H\|_{L^p(M)} |M|^{\frac{p-2}{2p}} \leq c_3 r$$

for some $r \in (0,1]$. If either

$$g < r \ \text{on } \partial M$$

or

$$g > r \ \text{on } \partial M \ \text{and } \inf_M g < \frac{3}{4}r,$$

then

$$g < r \ \text{on } \partial M \ \text{and } \sup_M g \le \frac{5}{4}r.$$

Proof. Let

$$\widetilde{g} = \begin{cases} r - g, & \text{if } g > r \text{ on } \partial M, \\ g - r, & \text{if } g < r \text{ on } \partial M, \end{cases}$$

so that $\widetilde{g} < 0$ on ∂M. We claim that

$$\widetilde{g} \le \frac{r}{4} \text{ on } M. \tag{7}$$

We first show how the lemma follows easily from (7) above. If (7) were true, then it remains to show that $\widetilde{g}$ must be in fact equal to $g - r$, because in that case

$$\widetilde{g} \le \frac{r}{4} \implies g \le \frac{5}{4}r.$$

Suppose that instead $\widetilde{g} = r - g$, i.e. $g > r$ on ∂M. Then (7) implies that $\widetilde{g} = r - g \le \frac{r}{4}$ on M and thus

$$g \ge \frac{3}{4}r \ \text{ on } M.$$

This contradicts the fact that $\inf_M g < \frac{3}{4}r$.

We now prove equation (7), i.e. that $\widetilde{g} \le \frac{r}{4}$ on M. We begin by defining the following sequence

$$r_0 = 0,\ r_1 = \frac{r}{2^3},\ \ldots,\ r_k = \sum_{i=1}^{k} \frac{r}{2^{i+2}} = r\frac{2^k - 1}{2^{k+2}},\ \ldots$$

for which we note that

$$0 = r_0 < r_1 < \cdots < r_k < \cdots < r/4 \text{ and } r_k - r_{k-1} = \frac{r}{2^{k+2}}.$$

Since $|\nabla \widetilde{g}| = |\nabla g| \le |A|$ (see for instance [11, Proof of Lemma 1]), the Jacobian of $\widetilde{g}$ is bounded by $|A|$ and thus applying the co-area formula [15, §10] we obtain that

$$\int_{r_{k-1}}^{r_k} |\Gamma_s| ds \le \int_{M_k} |A| \, d\mathcal{H}^2 \tag{8}$$

where, for any $s \in (r_{k-1}, r_k)$,

$$\Gamma_s = \{x \in M : \widetilde{g}(x) = s\} \text{ and } M_k = \{x \in M : r_{k-1} < \widetilde{g}(x) < r_k\}.$$

Applying Sard's Theorem, for each k we can pick $s_k \in (r_{k-1}, r_k)$, such that Γ_{s_k} is a collection of smooth Jordan curves and such that

$$|\Gamma_{s_k}| \leq \frac{2^{k+2}}{r} \int_{M_k} |A| \, d\mathcal{H}^2. \tag{9}$$

For each k let

$$U_k := \{x \in M : \widetilde{g}(x) > s_k\},$$

with the s_k's as above, and note that $U_k \subset M \setminus \partial M$, since on ∂M we have $\widetilde{g} < 0$. Furthermore

$$s_1 < s_2 < \cdots < s_k < \ldots \implies U_1 \supset U_2 \supset \cdots \supset U_k \supset \ldots$$

and $\lim\limits_{k\to\infty} s_k = r/4$. Let

$$U_\infty := \{x \in M : \widetilde{g}(x) \geq r/4\} = \bigcap_{k\in\mathbb{N}} U_k,$$

then, to prove (7) it suffices to show that

$$|U_\infty| = \left| \bigcap_{k\in\mathbb{N}} U_k \right| = \lim_{k\to\infty} |U_k| = 0. \tag{10}$$

That is because if claim (7) does not hold, then there would be a point $p \in M$ such that $\widetilde{g}(p) > \frac{r}{4}$ implying, since $\widetilde{g}$ is continuous, that $|U_\infty| > 0$.

In order to prove equation (10), let $G\colon S^2 \to \mathbb{R}$ denote the map

$$G((\nu_1, \nu_2, \nu_3)) = \sqrt{1 - \nu_3} = \frac{1}{\sqrt{2}} |(\nu_1, \nu_2, \nu_3) - e_3|$$

and let

$$\widetilde{G} = \begin{cases} r - G, & \text{if } g > r \text{ on } \partial M, \\ G - r, & \text{if } g < r \text{ on } \partial M. \end{cases}$$

Note that $\widetilde{g} = \widetilde{G} \circ \nu$ where ν is the Gauss map of M. Then

$$\nu(\partial U_k) \subset D_k := \left\{ (\nu_1, \nu_2, \nu_3) \in \mathbb{S}^2 : \widetilde{G}((\nu_1, \nu_2, \nu_3)) = s_k \right\}$$

and

$$\nu(U_k) \subset \Delta_k := \left\{ (\nu_1, \nu_2, \nu_3) \in \mathbb{S}^2 : \widetilde{G}((\nu_1, \nu_2, \nu_3)) > s_k \right\}.$$

Since $-K$ is the signed area magnification of the Gauss map, we have

$$\int_{U_k} (-K)\, d\mathcal{H}^2 = n|\Delta_k| \tag{11}$$

where $n \in \mathbb{Z}$ is the degree of the map ν. Therefore,

$$\int_{U_k} |K|\, d\mathcal{H}^2 \geq |n||\Delta_k|. \tag{12}$$

We claim that

$$|\Delta_k| \geq 2\pi \min\left\{\left(\frac{3}{4}r\right)^2, \frac{7}{16}\right\}. \tag{13}$$

In order to prove the claim, we need to discuss two separate cases depending on the definition of $\widetilde{g}$ and thus of $\widetilde{G}$.

Case 1: $\widetilde{g} = r - g$ and $\widetilde{G} = r - G$. In this case

$$D_k = \left\{(\nu_1, \nu_2, \nu_3) \in \mathbb{S}^2 : \nu_3 = 1 - (r - s_k)^2\right\},$$

$$\Delta_k = \left\{(\nu_1, \nu_2, \nu_3) \in \mathbb{S}^2 : \nu_3 > 1 - (r - s_k)^2\right\}.$$

Since $s_k < \frac{r}{4}$, this implies that Δ_k contains the upper spherical cap that has boundary $\nu_3 = 1 - \left(\frac{3}{4}r\right)^2$, whose area is $2\pi\left(\frac{3}{4}r\right)^2$. Therefore,

$$|\Delta_k| \geq 2\pi\left(\frac{3}{4}r\right)^2.$$

Case 2: $\widetilde{g} = g - r$ and $\widetilde{G} = G - r$. In this case

$$D_k = \left\{(\nu_1, \nu_2, \nu_3) \in \mathbb{S}^2 : \nu_3 = 1 - (r + s_k)^2\right\},$$

$$\Delta_k = \left\{(\nu_1, \nu_2, \nu_3) \in \mathbb{S}^2 : \nu_3 < 1 - (r + s_k)^2\right\}.$$

Since $s_k < \frac{r}{4}$ and $r \leq 1$, this implies that Δ_k contains the lower spherical cap that has boundary $\nu_3 = 1 - \left(\frac{5}{4}\right)^2$, whose area is $2\pi\left(2 - \left(\frac{5}{4}\right)^2\right)$. Therefore,

$$|\Delta_k| \geq 2\pi\left(2 - \left(\frac{5}{4}\right)^2\right) = 2\pi\frac{7}{16}.$$

Hence the claim (13) is true, that is

$$|\Delta_k| \geq 2\pi \min\left\{\left(\frac{3}{4}r\right)^2, \frac{7}{16}\right\}.$$

By the inequalities (12) and (13) and recalling the hypothesis of the lemma on $\int |A|^2\, d\mathcal{H}^2$, we have

$$2\pi \min\left\{\left(\frac{3}{4}r\right)^2, \frac{7}{16}\right\}|n| \le |\Delta_k||n| \le \int_{U_k} |K|\, d\mathcal{H}^2$$
$$\le \int_{U_k} \frac{|A|^2}{2}\, d\mathcal{H}^2 \le \frac{1}{2}\int_M |A|^2\, d\mathcal{H}^2 \le \frac{\pi r^2}{4},$$

which implies that $n = 0$, since $r \le 1$.

Now, since $n = 0$, equation (11) gives that

$$\int_{U_k} -K\, d\mathcal{H}^2 = 0.$$

Hence by the Gauss equation

$$\int_{U_k} |A|^2\, d\mathcal{H}^2 = \int_{U_k} H^2\, d\mathcal{H}^2. \tag{14}$$

Applying the isoperimetric inequality (1) with $F = U_k$ we obtain

$$|U_k|^{1/2} \le C\left(|\partial U_k| + \int_{U_k} |H|\, d\mathcal{H}^2\right)$$

and since $\partial U_k = \Gamma_{s_k}$, using (9) we get

$$|U_k|^{1/2} \le C\left(\frac{2^{k+2}}{r}\int_{M_k} |A|\, d\mathcal{H}^2 + \int_{U_k} |H|\, d\mathcal{H}^2\right)$$
$$\le C\left(\frac{2^{k+2}}{r}\int_{U_{k-1}} |A|\, d\mathcal{H}^2 + \int_{U_k} |H|\, d\mathcal{H}^2\right)$$
$$\le C\frac{2^{k+3}}{r}\int_{U_{k-1}} |A|\, d\mathcal{H}^2,$$

where we have used the facts $|H| \le 2|A|$, $M_k \subset U_{k-1}$, $U_k \subset U_{k-1}$ and $2 < \frac{2^{k+2}}{r}$, since $r \le 1$. Using Holder inequality and then squaring both sides of the inequality gives

$$|U_k| \le C_1 \frac{2^{2k}}{r^2}|U_{k-1}|\int_{U_{k-1}} |A|^2\, d\mathcal{H}^2$$

where $C_1 = (8C)^2$ is an absolute constant.

Applying (14) and Holder inequality we have

$$|U_k| \le C_1 2^{2k} r^{-2}|U_{k-1}|\int_{U_{k-1}} |H|^2\, d\mathcal{H}^2 \le C_1 2^{2k} r^{-2}\|H\|^2_{L^p(M)}|U_{k-1}|^{\frac{q+1}{q}}, \tag{15}$$

where q is such that $1/q + 2/p = 1$. By iterating (15) we obtain:

$$|U_k|^{\frac{q}{q+1}} \leq \left(C_1 r^{-2} \|H\|^2_{L^p(M)}\right)^{\frac{q}{q+1}} 4^{k\frac{q}{q+1}} |U_{k-1}|$$

$$\begin{aligned}
|U_k|^{\left(\frac{q}{q+1}\right)^2} &\leq \left(C_1 r^{-2} \|H\|^2_{L^p(M)}\right)^{\left(\frac{q}{q+1}\right)^2} 4^{k\left(\frac{q}{q+1}\right)^2} |U_{k-1}|^{\frac{q}{q+1}} \\
&\leq \left(C_1 r^{-2} \|H\|^2_{L^p(M)}\right)^{\left(\frac{q}{q+1}\right)+\left(\frac{q}{q+1}\right)^2} 4^{(k-1)\frac{q}{q+1}+k\left(\frac{q}{q+1}\right)^2} |U_{k-2}| \\
&\vdots \\
|U_k|^{\left(\frac{q}{q+1}\right)^{k-1}} &\leq \left(C_1 r^{-2} \|H\|^2_{L^p(M)}\right)^{\sum_{i=1}^{k-1}\left(\frac{q}{q+1}\right)^i} 4^{\sum_{i=1}^{k-1}(i+1)\left(\frac{q}{q+1}\right)^i} |U_1|
\end{aligned}$$

and letting $k \to \infty$ gives

$$\begin{aligned}
\lim_{k\to\infty} |U_k|^{\left(\frac{q}{q+1}\right)^{k-1}} &\leq \left(C_1 r^{-2} \|H\|^2_{L^p(M)}\right)^{\sum_{i=1}^{\infty}\left(\frac{q}{q+1}\right)^i} 4^{\sum_{i=1}^{\infty}(i+1)\left(\frac{q}{q+1}\right)^i} |U_1| \\
&\leq \left(C_1 r^{-2} \|H\|^2_{L^p(M)}\right)^q 4^{\alpha\frac{2-\alpha}{(1-\alpha)^2}} |U_1| = C_1^q 4^{\alpha\frac{2-\alpha}{(1-\alpha)^2}} \|H\|^{2q}_{L^p(M)} r^{-2q} |U_1|
\end{aligned}$$

where $\alpha = \frac{q}{q+1}$.

Using (15), with $k = 1$ and with U_{k-1} replaced by M we have

$$|U_1| \leq 4C_1 r^{-2} |M|^{\frac{q+1}{q}} \|H\|^2_{L^p(M)}$$

and thus

$$\lim_{k\to\infty} |U_k|^{\left(\frac{q}{q+1}\right)^{k-1}} \leq C_2 \|H\|^{2q+2}_{L^p(M)} |M|^{\frac{q+1}{q}} r^{-2-2q}, \tag{16}$$

where C_2 is a constant that depends solely on p. Assume that equation (10) is not true, namely assume that $|U_\infty| > 0$. Then, since $U_\infty = \bigcap_{k\in\mathbb{N}} U_k \subset U_k$ and $\frac{q}{q+1} < 1$, we have

$$\lim_{k\to\infty} |U_k|^{\left(\frac{q}{q+1}\right)^{k-1}} = 1$$

and using this in (16) we get

$$1 \leq C_2 \|H\|^{2q+2}_{L^p(M)} |M|^{\frac{q+1}{q}} r^{-2-2q}.$$

Therefore if in the hypotheses of the lemma we take c_3 to be

$$c_3 = \left(\frac{1}{2C_2}\right)^{\frac{1}{2(q+1)}} \tag{17}$$

then

$$\|H\|_{L^p(M)} |M|^{\frac{1}{2q}} r^{-1} \leq \left(\frac{1}{2C_2}\right)^{\frac{1}{2(q+1)}}$$

which in turn gives

$$1 \le C_2 \|H\|_{L^p(M)}^{2q+2} |M|^{\frac{q+1}{q}} r^{-2-2q} \le \frac{1}{2}.$$

This contradiction proves that actually $|U_\infty| = 0$. As we discussed before, this implies claim (7), that is $\widetilde{g} \le \frac{r}{4}$, which in turn implies the lemma and thus, taking c_3 as given by equation (17) finishes the proof. □

We are now ready to prove Theorem 1.3 in the introduction. It says that if the L^2 norm of $|A|$ and the L^p norm of H are sufficiently small, $p > 2$, then a geodesic ball is graphical away from its boundary. For convenience, we recall its statement.

Theorem 1.3. *There exist constants $c_1 > 0$ and $\beta \in (0, \frac{1}{2})$ such that the following holds. Given $p > 2$ there exists $c_2 = c_2(p)$ such that if $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$ is such that*

$$\int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2 \le c_1 r^2 \quad \textit{and} \quad \|H\|_{L^p(\mathcal{B}_R)} R^{\frac{p-2}{p}} \le c_2 r$$

for some $r \in [0, 1]$ then, after a rotation,

$$\sup_{\mathcal{B}_{\beta R}} g \le r.$$

Proof. Let $c_2 = \frac{c_3}{4\pi}$, where $c_3 = c_3(p)$ is the constant in Lemma 3.1. To prove this theorem we will show that there exists $s_0 \in [r/2, 4r/5]$ and $\beta \in (0, \frac{1}{2})$, such that if c_1 is small enough, then all the hypotheses of Lemma 3.1 are satisfied with $M = \mathcal{B}_{\beta R}$ and $r = s_0$.

After rotating the surface, we can assume that $\nu(x_0) = e_3$, i.e. $g(x_0) = 0$, where recall that x_0 is the center of the given geodesic ball $\mathcal{B}_R := \mathcal{B}_R(x_0)$. By Lemma 2.3,

$$\int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2 \le c_1 r^2 \le 2\pi c_1 \implies |\mathcal{B}_R| \le \pi R^2 (1 + c_1). \tag{18}$$

Note also that if $c_1 \le \pi/8$ then for any $\beta \in (0, 1]$ and any $s \in [r/2, 4r/5]$, we have

$$\int_{\mathcal{B}_{\beta R}} |A|^2 \, d\mathcal{H}^2 \le c_1 r^2 \le \frac{\pi}{8} 4 \left(\frac{r}{2}\right)^2 \le \frac{\pi}{2} s^2$$

and using (18) we also have

$$\begin{aligned} \|H\|_{L^p(\mathcal{B}_{\beta R})} |\mathcal{B}_{\beta R}|^{\frac{p-2}{2p}} &\le \|H\|_{L^p(\mathcal{B}_R)} |\mathcal{B}_R|^{\frac{p-2}{2p}} \le c_2 r R^{-\frac{p-2}{p}} |\mathcal{B}_R|^{\frac{p-2}{2p}} \\ &\le \frac{c_3}{4\pi} r R^{-\frac{p-2}{p}} R^{\frac{p-2}{p}} (\pi(1 + c_1))^{\frac{p-2}{2p}} \le \frac{c_3 r}{4\pi} (2\pi)^{\frac{p-2}{2p}} \\ &\le \frac{c_3}{4\pi} \frac{r}{2} 4\pi \le c_3 s. \end{aligned}$$

To apply Lemma 3.1 it remains to show that there exists $s_0 \in [r/2, 4r/5]$ and $\beta \in (0, 1]$, such that on $\partial \mathcal{B}_{\beta R}$ either $g > s_0$ or $g < s_0$. We obtain this by showing that we can find β and s_0 such that $\partial\mathcal{B}_{\beta R}$ consists of exactly one connected component and $g \neq s_0$ on $\partial\mathcal{B}_{\beta R}$. In fact, we will show that $\mathcal{B}_{\beta R}$ is homeomorphic to a disk.

Arguing exactly as we did in the proof of Lemma 3.1, equation (8), since $|\nabla g| \leq |A|$, the Jacobian of g is bounded by $|A|$ and thus applying the co-area formula in $\mathcal{B}_R$ for the function g, we get:

$$\begin{aligned}\int_{\frac{r}{2}}^{\frac{4r}{5}} |\Gamma_s| ds \leq \int_{M_r} |A|\, d\mathcal{H}^2 \leq |M_r|^{1/2} \left(\int_{M_r} |A|^2\, d\mathcal{H}^2\right)^{1/2} \\ \leq |\mathcal{B}_R|^{1/2} \left(\int_{\mathcal{B}_R} |A|^2\, d\mathcal{H}^2\right)^{1/2} \leq Rr\left(\pi c_1(1+c_1)\right)^{\frac{1}{2}}\end{aligned}$$

where $M_r = \left\{x \in \mathcal{B}_R : \frac{r}{2} < g(x) < \frac{4r}{5}\right\}$ and for any $s \in \left[\frac{r}{2}, \frac{4r}{5}\right]$, $\Gamma_s = \{x \in \mathcal{B}_R : g(x) = s\}$ and where we have used the inequality in (18).

By Sard's theorem, for almost all $s \in [r/2, 4r/5]$, Γ_s is collection of smooth, simple closed curves and we can pick $s_0 \in [r/2, 4r/5]$ such that

$$|\Gamma_{s_0}| \leq \frac{10}{3r} Rr\left(\pi c_1(1+c_1)\right)^{\frac{1}{2}} < 4R\left(\pi c_1(1+c_1)\right)^{\frac{1}{2}}.$$

Thus, if we let $\Delta := \{\rho \in (0, R) : \Gamma_{s_0} \cap \partial\mathcal{B}_\rho \neq \emptyset\}$ then

$$|\Delta| \geq R - 4R\left(\pi c_1(1+c_1)\right)^{\frac{1}{2}} = R\left(1 - 4\left(\pi c_1(1+c_1)\right)^{\frac{1}{2}}\right). \tag{19}$$

Note that $g|_{\partial\mathcal{B}_\rho} \neq s_0$ for any $\rho \in \Delta$. However, since $\partial\mathcal{B}_\rho$ consists of possibly more than one connected components, this does not imply that $g - s_0$ has a sign on $\partial\mathcal{B}_\rho$. If we can find ρ in Δ for which $\chi(\mathcal{B}_\rho) = 1$ then, for this ρ, $\mathcal{B}_\rho$ is homeomorphic to a disk, $\partial\mathcal{B}_\rho$ consists of a unique connected component and hence $g - s_0$ does have a sign on $\partial\mathcal{B}_\rho$. Recall that Lemma 2.2 states that

$$\int_{\mathcal{B}_R} |A|^2\, d\mathcal{H}^2 \leq \frac{1}{8C^2} \implies |T_1| \geq \frac{1}{32\pi C^2} R, \tag{20}$$

where C is the isoperimetric constant given in (1), and where

$$T_1 = \{\rho \in [0, R] : \chi(\mathcal{B}_\rho) = 1\}.$$

Let $\beta = \frac{1}{2(32\pi C^2)}$, then (20) becomes

$$|T_1| \geq 2\beta R.$$

If we take c_1 sufficiently small such that

$$4\left(\pi c_1(1+c_1)\right)^{\frac{1}{2}} \leq \beta$$

then the hypothesis and thus the implication of (20) holds and (19) becomes

$$|\Delta| \geq R\,(1-\beta)\,.$$

Therefore for some $\gamma \geq \beta$ we have that $\gamma R \in \Delta \cap T_1$, namely $\mathcal{B}_{\gamma R}$ is homeomorphic to a disk, $\partial \mathcal{B}_{\gamma R}$ consists of one connected component and $g \neq s_0$ on $\partial \mathcal{B}_{\gamma R}$. Hence, by applying Lemma 3.1 to $\mathcal{B}_{\gamma R}$ we have that

$$\sup_{\mathcal{B}_{\beta R}} g \leq \sup_{\mathcal{B}_{\gamma R}} g \leq \frac{5}{4} s_0 \leq r.$$

This finishes the proof of the theorem. $\square$

From the above theorem, an extrinsic version of the same theorem follows.

Corollary 1.4. *Let M be an orientable surface containing the origin with $\partial M \subset \partial B_R(0)$,*

$$\int_M |A|^2 \, d\mathcal{H}^2 \leq c_1 r^2 \quad \text{and} \quad \|H\|_{L^p(M \cap B_R)} R^{\frac{p-2}{p}} \leq c_2 r$$

for some $r \in [0, \frac{1}{\sqrt{3}}]$, $p > 2$. Then, if M_R is a connected component of $M \cap B_{\frac{\beta R}{2}}$ containing the origin, after a rotation

$$\sup_{M_R} g \leq r,$$

where the constants c_1, c_2 and β are the constants in Theorem 1.3.

Proof. Let M_R be a connected component of $M \cap B_{\frac{\beta R}{2}}$ containing the origin and let $\mathcal{B}_R$ be "the" geodesic ball of radius R centered at the origin. Note that since M is not assumed to be embedded, the pre-image of the origin in $\mathbb{R}^3$ may consist of several points in M. Thus, by $\mathcal{B}_R$ we indicate a geodesic ball of radius R centered at one of those pre-images related to M_R. By the previous theorem, after a rotation,

$$\sup_{\mathcal{B}_{\beta R}} g \leq r$$

and since $r \leq \frac{1}{\sqrt{3}}$, this gives that $\mathcal{B}_{\beta R}$ contains a graph over a domain $\Omega \subset \mathbb{R}^2$ with

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq \left(\frac{\beta R}{2}\right)^2 \right\} \subset \Omega$$

(see Lemma 5.2) and hence $\partial\Omega \cap B_{\frac{\beta R}{2}} = \emptyset$, which finishes the proof of the corollary. $\square$

In the next corollary we prove that if the L^p norm of the mean curvature is bounded, $p > 2$, then, if the L^2 norm of $|A|$ is sufficiently small, a geodesic ball is graphical away from its boundary.

Corollary 3.2. *There exists a constant $\gamma \in (0, \frac{1}{4})$ such that the following holds. Given any $p > 2$ and $K > 0$, there exists $\varepsilon = \varepsilon(p, K)$ such that if $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$,*

$$\int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2 \leq \varepsilon^4 r^2 \quad \text{and} \quad \|H\|_{L^p(\mathcal{B}_R)} R^{\frac{p-2}{p}} \leq Kr$$

for some $r \in [0, 1]$ then, after a rotation,

$$\sup_{\mathcal{B}_{\gamma R}} g \leq \frac{3}{2} r.$$

Proof. Since the statement is scale invariant, it suffices to prove it for $R = 1$ and, after possibly rotating, we assume that $g(x_0) = 0$, where recall that x_0 is the center of the given geodesic ball $\mathcal{B}_R := \mathcal{B}_R(x_0)$.

As in the proof of Theorem 1.3, if ε is taken sufficiently small (e.g. $\varepsilon \leq c_1$, where c_1 is as in Theorem 1.3) there exists $\gamma \in \left[\frac{\beta}{2}, 1 - \frac{\beta}{2}\right]$ such that $\mathcal{B}_\gamma$ is a disk and either $g > r$ or $g < r$ on $\partial \mathcal{B}_\gamma$; β is the universal constant as in Theorem 1.3. Recall that in that step of the proof of Theorem 1.3 the estimate on the L^p norm of H is not used. Recall also that from Lemmas 2.1 and 2.3

$$\frac{1}{16C^2} \rho^2 \leq |\mathcal{B}_\rho| \leq \rho^2 (\pi + r^2 \varepsilon^2) \, , \; \forall \rho \leq 1. \tag{21}$$

(Note that Lemma 2.1 is applicable provided that ε is small enough, e.g $\varepsilon \leq \frac{1}{2C}$, where C is the isoperimetric constant as in (1).)

Let

$$\widetilde{g} = \begin{cases} r - g, & \text{if } g > r \text{ on } \partial \mathcal{B}_\gamma, \\ g - r, & \text{if } g < r \text{ on } \partial \mathcal{B}_\gamma, \end{cases}$$

so that $\widetilde{g} < 0$ on $\partial \mathcal{B}_\gamma$. Since $|\nabla \widetilde{g}| \leq |A|$, using the coarea formula we get

$$\int_{r/8}^{r/4} |\Gamma_s| ds \leq \int_{\mathcal{B}_\gamma} |A| \, d\mathcal{H}^2,$$

where $\Gamma_s = \{x \in \mathcal{B}_\gamma : \widetilde{g}(x) = s\}$ and applying Sard's Theorem, there exists $s_0 \in [r/8, r/4]$ such that Γ_{s_0} is a collection of smooth Jordan curves satisfying

$$|\Gamma_{s_0}| \leq \frac{8}{r} \int_{\mathcal{B}_\gamma} |A| \, d\mathcal{H}^2 \leq \frac{8}{r} \left(\int_{\mathcal{B}_1} |A|^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2}} |\mathcal{B}_1|^{\frac{1}{2}} \leq 16 \varepsilon^2,$$

where in the last inequality we have used (21) with $\rho = 1$, which implies that $|\mathcal{B}_1| \leq 4$, when ε small enough, e.g. $\varepsilon \leq \frac{1}{2}$.

Let

$$U = \{x \in \mathcal{B}_\gamma : \widetilde{g}(x) > s_0\}.$$

Since $\widetilde{g} < 0$ on $\partial\mathcal{B}_\gamma$, then $\Gamma_{s_0} \cap \partial\mathcal{B}_\gamma = \emptyset$ and thus $\partial U = \Gamma_{s_0}$. By applying the isoperimetric inequality (1) to U we obtain

$$\begin{aligned}
|U|^{\frac{1}{2}} &\le C\left(|\partial U| + \int_U |H|\, d\mathcal{H}^2\right) = C\left(|\Gamma_{s_0}| + \int_U |H|\, d\mathcal{H}^2\right)\\
&\le C\left(\frac{8}{r}\int_{\mathcal{B}_1} |A|\, d\mathcal{H}^2 + |\mathcal{B}_1|^{\frac{1}{2}}\left(\int_U |H|^2\, d\mathcal{H}^2\right)^{\frac{1}{2}}\right)\\
&\le C\left(\frac{8}{r}|\mathcal{B}_1|^{\frac{1}{2}}\left(\int_{\mathcal{B}_1} |A|^2\, d\mathcal{H}^2\right)^{\frac{1}{2}} + 2|\mathcal{B}_1|^{\frac{1}{2}}\left(\int_{\mathcal{B}_1} |A|^2\, d\mathcal{H}^2\right)^{\frac{1}{2}}\right)\\
&\le C\frac{10}{r}|\mathcal{B}_1|^{\frac{1}{2}}\left(\int_{\mathcal{B}_1} |A|^2\, d\mathcal{H}^2\right)^{\frac{1}{2}}
\end{aligned}$$

hence

$$|U| \le \frac{(10C)^2}{r^2}|\mathcal{B}_1|\varepsilon^4 r^2 \le 4(10C)^2\varepsilon^4. \tag{22}$$

Let

$$\widetilde{U} = \{x \in \mathcal{B}_\gamma : \widetilde{g}(x) > 2s_0\}.$$

We claim that $\widetilde{U} = \emptyset$. We first show how the corollary with $\gamma = \frac{\beta}{2}$ follows from this claim. If $\widetilde{U}$ is empty then $\widetilde{g} \le 2s_0$ on $\mathcal{B}_\gamma$. If $\widetilde{g} = g - r$ then

$$g - r \le 2s_0 \le \frac{r}{2} \implies g \le \frac{3}{2}r.$$

Thus, it remains to show that the case $\widetilde{g} = r - g$ cannot occur. If $\widetilde{g} = r - g$ and since $\widetilde{g} \le 2s_0$ then

$$r - g \le 2s_0 \le \frac{r}{2} \implies g \ge \frac{r}{2}.$$

However, since $g(x_0) = 0$ this is a contradiction.

Thus it suffices to show that $\widetilde{U} = \emptyset$. Arguing by contradiction, suppose that $\widetilde{U} \ne \emptyset$, let $x \in \widetilde{U} \subset U$ and consider the geodesic ball $\mathcal{B}_\delta(x)$, for some $\delta > 0$. Since $x \in U \subset \mathcal{B}_\gamma$, for $\delta \le \frac{\beta}{2} \le 1 - \gamma$, where recall that β is the universal constant as in Theorem 1.3 and $\gamma \in \left[\frac{\beta}{2}, 1 - \frac{\beta}{2}\right]$, we have

$$\mathcal{B}_\delta(x) \subset \mathcal{B}_1.$$

Suppose that $\mathcal{B}_\delta(x) \subset U$, then by (21), with ρ replaced by δ and using also (22) we get

$$\frac{1}{16C^2}\delta^2 \le |\mathcal{B}_\delta(x)| \le |U| \le 4(10C)^2\varepsilon^4 \implies \frac{\varepsilon^4}{\delta^2} \ge \frac{1}{80^2C^4},$$

where C is the isoperimetric constant given in (1). Therefore, taking

$$\varepsilon < \min\left\{\frac{1}{80C^2}, \frac{\beta}{2}, c_1\right\},$$

where c_1 is as in Theorem 1.3 and then choosing $\delta = \varepsilon$ in the previous discussion, implies that

$$\mathcal{B}_\delta(x) \not\subset U.$$

In other words, if ε is sufficiently small, points in $\widetilde{U}$ are still intrinsically close to points in $\mathcal{B}_\gamma$ where $\widetilde{g} \leq s_0$. That is for each $x \in \widetilde{U}$, there exists $y_0 \in \mathcal{B}_\varepsilon(x)$ such that $\widetilde{g}(y_0) \leq s_0$.

Let ε be small enough so that $\varepsilon \leq \frac{\beta^2}{4} \leq \frac{\beta}{2}(1-\gamma)$, then for any $x \in U \subset \mathcal{B}_\gamma$ we have that

$$\mathcal{B}_{\varepsilon\left(\frac{\beta}{2}\right)^{-1}}(x) \subset \mathcal{B}_1.$$

So far we have not used the bound on the L^p norm of H and that is what we are going to do next. Clearly,

$$\left(\frac{2\varepsilon}{\beta}\right)^{\frac{p-2}{p}} \|H\|_{L^p(\mathcal{B}_{2\beta^{-1}\varepsilon}(x))} \leq \left(\frac{2\varepsilon}{\beta}\right)^{\frac{p-2}{p}} \|H\|_{L^p(\mathcal{B}_1)} \leq \left(\frac{2\varepsilon}{\beta}\right)^{\frac{p-2}{p}} Kr.$$

Therefore, if

$$\left(\frac{2\varepsilon}{\beta}\right)^{\frac{p-2}{p}} Kr \leq c_2 \frac{r}{10}$$

and $\varepsilon^4 r^2 \leq c_1 \left(\frac{r}{10}\right)^2$, where c_1, c_2 are the constants given by Theorem 1.3, then applying Theorem 1.3 with $\mathcal{B}_R$ replaced by $\mathcal{B}_{2\beta^{-1}\varepsilon}(x)$, gives that for any $x \in \widetilde{U}$,

$$\sup_{y \in \mathcal{B}_{2\varepsilon}(x)} |\nu(y) - \nu(x)| \leq \sqrt{2}\frac{r}{10}.$$

This, together with the previous observation, gives that for any $x \in \widetilde{U}$ there exists $y_0 \in \mathcal{B}_\varepsilon(x)$ such that

$$\widetilde{g}(y_0) \leq s_0 \quad \text{and} \quad |\nu(y_0) - \nu(x)| \leq \sqrt{2}\frac{r}{10}.$$

In order to prove that $\widetilde{U}$ is empty, we show that the above observations lead to a contradiction. In order to do this, we have to discuss two separate cases, depending on what $\widetilde{g}$ is. The arguments are almost the same and the idea is simple: since the unit normal vector at x and the unit normal vector at y_0 are close, if the unit normal vector at y_0 is close to e_3 then so is the unit normal vector at x.

Case 1: $\widetilde{g} = g - r$. Let $x \in \widetilde{U}$ then, in addition to $|\nu(y_0) - \nu(x)| \le \sqrt{2}\frac{r}{10}$ we have that

$$\widetilde{g}(y_0) \le s_0 \implies g(y_0) - r \le s_0 \implies |\nu(y_0) - e_3| \le \sqrt{2}(r + s_0)$$

$$\widetilde{g}(x) \ge 2s_0 \implies g(x) - r \ge 2s_0 \implies |\nu(x) - e_3| \ge \sqrt{2}(r + 2s_0)$$

Therefore, combining the three inequalities we obtain

$$\sqrt{2}(r+2s_0) \le |\nu(x) - e_3| \le |\nu(y_0) - \nu(x)| + |\nu(y_0) - e_3| \le \sqrt{2}\left(\frac{r}{10} + r + s_0\right),$$

that is

$$s_0 \le \frac{r}{10}.$$

Since $s_0 \ge \frac{r}{8}$ this leads to a contradiction.

Case 2: $\widetilde{g} = r - g$. Let $x \in \widetilde{U}$ then, in addition to $|\nu(y_0) - \nu(x)| \le \sqrt{2}\frac{r}{10}$ we have that

$$\widetilde{g}(y_0) \le s_0 \implies r - g(y_0) \le s_0 \implies |\nu(y_0) - e_3| \ge \sqrt{2}(r - s_0)$$

$$\widetilde{g}(x) \ge 2s_0 \implies r - g(x) \ge 2s_0 \implies |\nu(x) - e_3| \le \sqrt{2}(r - 2s_0)$$

Therefore, combining the three inequalities we obtain

$$\sqrt{2}(r - s_0) \le |\nu(y_o) - e_3| \le |\nu(y_0) - \nu(x)| + |\nu(x) - e_3| \le \sqrt{2}\left(\frac{r}{10} + r - 2s_0\right),$$

that is

$$s_0 \le \frac{r}{10}.$$

Since $s_0 \ge \frac{r}{8}$ this leads to a contradiction.

We have proven that, independently of the definition of $\widetilde{g}$, the set $\widetilde{U}$ is empty. This, as shown previously, finishes the proof of the corollary. $\square$

4 Graph representation in terms of $\|A\|_{L^2}$, when $\|H\|_{L^p}$ is small

In this section we use results from the previous sections to prove that an embedded geodesic disk with bounded L^2 norm of $|A|$ and sufficiently small L^p norm of the mean curvature, $p > 2$, is graphical away from its boundary. This is related to previous results by Colding-Minicozzi for minimal surfaces [4], see also [2].

Definition 4.1. *Let M be a simply-connected surface embedded in $\mathbb{R}^3$. For any $x \in M$ and $R > 0$ such that $\mathcal{B}_R(x) \subset M \setminus \partial M$, $\partial\mathcal{B}_R(x)$ has a component that is the boundary of a disk in M that contains $\mathcal{B}_R(x)$. We denote this disk by $\mathcal{B}^*_R(x)$.*

The following lemma shows that given an embedded geodesic ball $\mathcal{B}_R$ in a simply-connected surface, if $\mathcal{B}_R$ has small L^2 norm of $|A|$ away from the origin and also $\mathcal{B}_R^*$ has sufficiently small L^p norm of H, then this geodesic ball is graphical away from its boundary.

Lemma 4.2. *Given $K \geq 0$ and $N \geq 20$ there exists $\varepsilon_1 = \varepsilon_1(K,N) > 0$ such that for any $p > 2$ the following holds. Let M be a simply-connected surface embedded in $\mathbb{R}^3$ containing the origin and let $\mathcal{B}_N := \mathcal{B}_N(0) \subset M \setminus \partial M$ be such that*

$$\int_{\mathcal{B}_N} |A|^2 \, d\mathcal{H}^2 \leq K, \quad \int_{\mathcal{B}_N \setminus \mathcal{B}_1} |A|^2 \, d\mathcal{H}^2 \leq c_1(\varepsilon_1 r)^2 \quad \textit{and}$$

$$(16C^2|\mathcal{B}_N^*|)^{\frac{p-2}{2p}} \left(\int_{\mathcal{B}_N^*} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq c_2 \varepsilon_1 r,$$

for some $r \in [0, \frac{1}{4}]$, where $c_1, c_2 = c_2(p)$ are as in Theorem 1.3 *and where C is the isoperimetric constant as in* (1). *Then, after a rotation,*

(i)

$$\sup_{\mathcal{B}_{N-1} \setminus \mathcal{B}_2^*} g \leq r,$$

(ii)

$$\int_{\mathcal{B}_2^*} |A|^2 \, d\mathcal{H}^2 \leq (c_2 \varepsilon_1 r)^2 + 24\pi \varepsilon_1 r \left(\frac{2K + 4(N-3)}{\beta} + 1 + c_1 \right),$$

where β as in Theorem 1.3 *and*

(iii)

$$\sup_{\mathcal{B}_2^*} g \leq \frac{5}{4}\sqrt{r}.$$

Proof. Note that

$$\left(\int_{\mathcal{B}_N} |H|^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2}} \leq \left(\int_{\mathcal{B}_N^*} |H|^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2}} \leq |\mathcal{B}_N^*|^{\frac{p-2}{2p}} \left(\int_{\mathcal{B}_N^*} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq \frac{c_2 \varepsilon_1 r}{(16C^2)^{\frac{p-2}{2p}}}.$$

Hence, if $\varepsilon_1 \leq \frac{1}{2c_2 C}$ then $\frac{c_2 \varepsilon_1 r}{(16C^2)^{\frac{p-2}{2p}}} \leq \frac{1}{2C}$, and we can apply Lemma 2.1, which gives

$$N^2 \leq 16C^2|\mathcal{B}_N| \leq 16C^2|\mathcal{B}_N^*|.$$

Therefore,

$$N^{\frac{p-2}{p}} \left(\int_{\mathcal{B}_N} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq (16C^2|\mathcal{B}_N^*|)^{\frac{p-2}{2p}} \left(\int_{\mathcal{B}_N^*} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq c_2 \varepsilon_1 r$$

Furthermore, for any $x \in \mathcal{B}_{N-1} \setminus \mathcal{B}_2$, we have that $\mathcal{B}_1(x) \subset \mathcal{B}_N \setminus \mathcal{B}_1$ and thus, by the previous discussion and the assumptions on $|A|^2$, we note that the hypotheses of Theorem 1.3 are satisfied with $\mathcal{B}_R(x_0)$ replaced by $\mathcal{B}_1(x)$ and with r replaced by $\varepsilon_1 r$. Applying Theorem 1.3 gives then that

$$\frac{1}{\sqrt{2}}|\nu(y) - \nu(x)| \leq \varepsilon_1 r\,,\ \forall y \in \mathcal{B}_\beta(x)\,,\ \forall x \in \mathcal{B}_{N-1} \setminus \mathcal{B}_2 \tag{23}$$

with β as in Theorem 1.3. Since $\mathcal{B}_{N-1} \setminus \mathcal{B}_2^* \subset \mathcal{B}_{N-1} \setminus \mathcal{B}_2$, by using the triangle inequality and (23), we obtain the following estimate: for any $p, q \in \mathcal{B}_{N-1} \setminus \mathcal{B}_2^*$ let $\gamma \subset \mathcal{B}_{N-1} \setminus \mathcal{B}_2^*$ be a curve connecting p and q, then

$$\frac{1}{\sqrt{2}}|\nu(p) - \nu(q)| \leq \left(\frac{2|\gamma|}{\beta} + 1\right)\varepsilon_1 r, \tag{24}$$

where recall that $|\gamma|$ denotes the length of the curve γ. To see this, let $\{p_i\}_{i=0}^m$ be points on γ such that $p_0 = p$, $p_m = q$ and $\mathrm{dist}_\Sigma(p_i, p_{i+1}) \leq \beta/2$. Note that we can do this with $m = \left[\frac{2|\gamma|}{\beta}\right] + 1$ points. Then,

$$\frac{1}{\sqrt{2}}|\nu(p_i) - \nu(p_{i+1})| \leq \varepsilon_1 r\,,\ \forall i = 0, 1, \dots m-1 \implies$$

$$\frac{1}{\sqrt{2}}|\nu(p) - \nu(q)| \leq m\varepsilon_1 r \leq \left(\frac{2|\gamma|}{\beta} + 1\right)\varepsilon_1 r$$

Thus, in order to prove *(i)* of the lemma, it remains to bound the diameter of $\mathcal{B}_{N-1} \setminus \mathcal{B}_2^*$. By Lemma 2.3 we have

$$|\partial \mathcal{B}_2^*| \leq |\partial \mathcal{B}_2| \leq 4\pi + \int_{\mathcal{B}_2} |A|^2 \, d\mathcal{H}^2 \leq 4\pi + K.$$

Without loss of generality, let us assume that $4\pi \leq K$ and thus

$$|\partial \mathcal{B}_2^*| \leq 2K.$$

This implies that any two points in $\mathcal{B}_{N-1} \setminus \mathcal{B}_2^*$ can be connected by a curve $\gamma \subset \mathcal{B}_{N-1} \setminus \mathcal{B}_2^*$ such that

$$|\gamma| \leq \frac{1}{2}|\partial \mathcal{B}_2^*| + 2((N-1) - 2) \leq K + 2(N-3). \tag{25}$$

Finally, combining (24) and (25), we have that for any $p, q \in \mathcal{B}_{N-1} \setminus \mathcal{B}_2^*$

$$\frac{1}{\sqrt{2}}|\nu(p) - \nu(q)| \leq \left(\frac{2K + 4(N-3)}{\beta} + 1\right)\varepsilon_1 r.$$

Taking

$$\varepsilon_1 \leq \left(\frac{2K + 4(N-3)}{\beta} + 1\right)^{-1}$$

and applying a rotation finishes the proof of *(i)* in the lemma. In fact, by letting

$$\delta = \left(\frac{2K + 4(N-3)}{\beta} + 1\right)\varepsilon_1 \leq 1,$$

we have that for any $p, q \in \mathcal{B}_{N-1} \setminus \mathcal{B}_2^*$

$$\frac{1}{\sqrt{2}}|\nu(p) - \nu(q)| \leq \delta r. \tag{26}$$

which implies that, after possibly applying a rotation, $\mathcal{B}_{N-1} \setminus \mathcal{B}_2^*$ is locally graphical over the plane $\{x_3 = 0\}$ with the norm of the gradient bounded by $3\delta r$ (cf. Lemma 5.2).

Part *(ii)* of the lemma states that the L^2 norm of $|A|$ is small on $\mathcal{B}_2^*$. We intend to show this by using the Gauss-Bonnet theorem together with our bound on the L^p norm of the mean curvature. To that end, we need to find a curve bounding a disk containing $\mathcal{B}_2^*$ and which has small total geodesic curvature.

We begin by showing that the projection of $\partial(\mathcal{B}_{(N+1)/2} \cup \mathcal{B}_2^*)$ on the plane $\{x_3 = 0\}$ is away from the origin. In particular, $\partial(\mathcal{B}_{(N+1)/2} \cup \mathcal{B}_2^*)$ is extrinsically distant from the origin.

Claim 4.3. *Let $C_\rho := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \leq \rho^2\}$ then*

$$\partial(\mathcal{B}_{(N+1)/2} \cup \mathcal{B}_2^*) \cap \partial C_{\frac{N-11}{4}} = \emptyset.$$

In particular $\partial\mathcal{B}_2^*$ lies inside C_2 and $\partial(\mathcal{B}_{(N+1)/2} \cup \mathcal{B}_2^*)$ outside $C_{\frac{N-11}{4}}$.

Proof of Claim 4.3. Given $x \in \partial\mathcal{B}_{\frac{N+1}{2}} \setminus \mathcal{B}_2^*$, consider the geodesic ball $\mathcal{B}_{\frac{N-3}{2}}(x)$. Note that $\partial\mathcal{B}_2^*$ is clearly contained in C_2 and that by our choice of radii, there exists at least one point $p \in \partial\mathcal{B}_{\frac{N-3}{2}}(x) \cap \partial\mathcal{B}_2^*$. Since $\mathcal{B}_{\frac{N-3}{2}}(x) \subset \mathcal{B}_{N-1} \setminus \mathcal{B}_2^*$, by (26) we have that $\mathcal{B}_{\frac{N-3}{2}}(x)$ is locally graphical over the plane $\{x_3 = 0\}$ with norm of the gradient bounded by $3\delta r$. In particular, if we let

$$\Pi\colon \mathbb{R}^3 \to \{x_3 = 0\}$$

be the projection to the plane $\{x_3 = 0\}$, then $\mathcal{B}_{\frac{N-3}{2}}(x)$ contains a graph over the disk in the plane $\{x_3 = 0\}$ centered at $\Pi(x)$ and of radius $\frac{N-3}{2\sqrt{1+(3\delta r)^2}}$ (cf. Lemma 5.2). This implies that for any $q \in \partial\mathcal{B}_{\frac{N-3}{2}}(x)$,

$$\begin{aligned}
|\Pi(x) - \Pi(q)| \geq& \frac{N-3}{2\sqrt{1+(3\delta r)^2}} - \left(\frac{N-3}{2} - \frac{N-3}{2\sqrt{1+(3\delta r)^2}}\right) \\
=& \frac{N-3}{\sqrt{1+(3\delta r)^2}} - \frac{N-3}{2}.
\end{aligned}$$

The above inequality holds because if γ is a geodesic connecting q and x of length $\frac{N-3}{2}$ then, by the previous discussion, there exists $y \in \gamma$ such that

$$|\Pi(y) - \Pi(x)| = \frac{N-3}{2\sqrt{1+(3\delta r)^2}}$$

and then the intrinsic distance between y and q is at most $\frac{N-3}{2} - \frac{N-3}{2\sqrt{1+(3\delta r)^2}}$.

Finally, since the above inequality holds with q replaced by any

$$p \in \partial\mathcal{B}_{\frac{N-3}{2}}(x) \cap \partial\mathcal{B}^*_2 \neq \emptyset$$

and because for such a p the inequality $|\Pi(p)| \leq 2$ holds, we have that

$$|\Pi(x)| \geq \frac{N-3}{\sqrt{1+(3\delta r)^2}} - \frac{N-3}{2} - 2.$$

Since $\delta \leq 1$, $N \geq 20$ and $r \leq 1/4$,

$$\frac{N-3}{\sqrt{1+(3\delta r)^2}} - \frac{N-3}{2} - 2 \geq \frac{3}{10}(N-3) - 2 \geq \frac{N-11}{4}.$$

This finishes the proof of the claim. $\square$

By the above claim and since $\mathcal{B}_{(N+1)/2} \setminus \mathcal{B}^*_2$ is embedded and locally a graph over the plane $\{x_3 = 0\}$, we have that $\partial C_{\frac{N-11}{4}} \cap (\mathcal{B}_{(N+1)/2} \setminus \mathcal{B}^*_2)$ is the union of simple closed curves that are graphs over

$$S_{\frac{N-11}{4}} = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = \left(\frac{N-11}{4}\right)^2\right\}.$$

By elementary topological arguments, there exists a component Γ_0 of $\partial C_{\frac{N-11}{4}} \cap (\mathcal{B}_{(N+1)/2} \setminus \mathcal{B}^*_2)$ such that Γ_0 bounds a disk in M containing $\mathcal{B}^*_2$. To see this, note that otherwise it would be possible to connect $\partial(\mathcal{B}_{(N+1)/2} \cup \mathcal{B}^*_2)$ with $\partial\mathcal{B}^*_2$ without intersecting $\partial C_{\frac{N-11}{4}}$, which is clearly a contradiction since the former boundary is outside $C_{\frac{N-11}{4}}$ while the latter boundary is inside $C_{\frac{N-11}{4}}$. Note that for each $x \in \Gamma_0$, we have that

$$\mathcal{B}_{\frac{N-19}{4}}(x) \subset \mathcal{B}_{N-1} \setminus \mathcal{B}^*_2.$$

Hence, using (26) and applying Lemma 5.2 in each of the geodesic balls $\mathcal{B}_{\frac{N-19}{4}}(x)$, we conclude that there exists a "thick" neighborhood of Γ_0 in M that can be written as a graph over the plane $\{x_3 = 0\}$. Namely there exist $b > \frac{N-11}{4} > a > 0$ such that if Ω denotes the annulus $\{(x_1, x_2) : a^2 \leq x_1^2 + x_2^2 \leq b^2\}$ then there exists a function

$$u : \Omega \to M$$

such that the following holds: the curve Γ_0 is contained in the graph of u, $\Gamma_0 \subset u(\Omega)$, the gradient of u satisfies $|Du| \leq 3\delta r$ and for a, b we have that

$$\begin{aligned} b - a &= \frac{N-19}{4} \cdot \frac{1}{\sqrt{1+(3\delta r)^2}} \ , \\ b = \frac{N-11}{4} &+ \frac{N-19}{4} \cdot \frac{1}{\sqrt{1+(3\delta r)^2}} \ , \quad a > 2. \end{aligned}$$

Now we note that

$$\int_{u(\Omega)} |A|^2 \, d\mathcal{H}^2 \leq \int_{\mathcal{B}_N \setminus \mathcal{B}_1} |A|^2 \, d\mathcal{H}^2 \leq c_1(\varepsilon_1 r)^2$$

and thus we can apply Lemma 5.1, with r and ε replaced by $3\delta r$ and $c_1(\varepsilon_1 r)^2$ respectively, to conclude that for some $\rho \in (a, b)$

$$\int_{u(S_\rho)} k \, ds \leq 2\pi \left(1 + 3\sqrt{2}\delta r + \left(\frac{2c_1(\varepsilon_1 r)^2 b}{b-a} \right)^{\frac{1}{2}} \right)$$

where k is the curvature of $u(S_\rho)$. Using now (4), we have that

$$\frac{b}{b-a} \leq \frac{N-11}{N-19} \cdot \sqrt{1+(3\delta r)^2} + 1 \leq \frac{N-11}{N-19} \cdot \frac{5}{4} + 1 \leq 13,$$

where we have used that $N \geq 20$, $\delta \leq 1$ and $r \leq \frac{1}{4}$. Thus we get

$$\int_{u(S_\rho)} k \, ds \leq 2\pi \left(1 + 3\sqrt{2}\delta r + \left(26 c_1 (\varepsilon_1 r)^2 \right)^{\frac{1}{2}} \right) \tag{27}$$

Let $\Gamma = u(S_\rho)$. Then, by construction Γ bounds a disk Δ that contains $\mathcal{B}_2^*$. Let k_g denote the geodesic curvature of Γ. Using the Gauss Bonnet theorem we have that

$$2\pi - \int_\Gamma k_g \, ds = \int_\Delta K_\Sigma \, d\mathcal{H}^2 = \frac{1}{2} \int_\Delta (H^2 - |A|^2) \, d\mathcal{H}^2. \tag{28}$$

Since

$$\left(\int_{\mathcal{B}_N^*} |H|^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2}} \leq |\mathcal{B}_N^*|^{\frac{p-2}{2p}} \left(\int_{\mathcal{B}_N^*} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq \frac{c_2 \varepsilon_1 r}{(16C^2)^{\frac{p-2}{2p}}} \leq c_2 \varepsilon_1 r,$$

using equation (28), $|k_g| \leq k$ and (27) gives

$$\begin{aligned} \int_{\mathcal{B}_2^*} |A|^2 \, d\mathcal{H}^2 \leq \int_\Delta |A|^2 \, d\mathcal{H}^2 &\leq -4\pi + \int_\Delta H^2 \, d\mathcal{H}^2 + 2 \int_\Gamma k_g \, ds \\ &\leq -4\pi + \int_{\mathcal{B}_N^*} |H|^2 \, d\mathcal{H}^2 + 2 \int_\Gamma k \, ds \\ &\leq -4\pi + (c_2 \varepsilon_1 r)^2 + 4\pi \left(1 + 6\delta r + 6 c_1 \varepsilon_1 r \right). \end{aligned}$$

Finally, since $\delta = \varepsilon_1 \left(\frac{2K+4(N-3)}{\beta} + 1\right)$, we have

$$\int_{\mathcal{B}_2^*} |A|^2 \, d\mathcal{H}^2 \leq (c_2\varepsilon_1 r)^2 + 24\pi\varepsilon_1 r \left(\frac{2K + 4(N-3)}{\beta} + 1 + c_1\right).$$

This finishes the proof of *(ii)* in the lemma.

The proof of *(iii)* is a simple consequence of *(ii)* and Lemma 3.1. Let Δ be the previously defined disk, see equation (28). The disk Δ contains $\mathcal{B}_2^*$ and

$$\begin{aligned}
\int_{\Delta} |A|^2 \, d\mathcal{H}^2 \leq & \int_{\mathcal{B}_2^*} |A|^2 \, d\mathcal{H}^2 + \int_{\mathcal{B}_N \setminus \mathcal{B}_1} |A|^2 \, d\mathcal{H}^2 \\
\leq & (c_2\varepsilon_1 r)^2 + 24\pi\varepsilon_1 r \left(\frac{2K + 4(N-3)}{\beta} + 1 + c_1\right) + c_1(\varepsilon_1 r)^2 \\
\leq & (\varepsilon_1 r)^2 \left(c_2^2 + c_1\right) + 24\pi\varepsilon_1 r \left(\frac{2K + 4(N-3)}{\beta} + 1 + c_1\right) \\
\leq & r\varepsilon_1 \left(c_2^2 + c_1 + 24\pi \left(\frac{2K + 4(N-3)}{\beta} + 1 + c_1\right)\right).
\end{aligned}$$

Moreover, since $\Delta \subset \mathcal{B}_N^*$ we have that

$$\|H\|_{L^p(\Delta)} |\Delta|^{\frac{p-2}{2p}} \leq \|H\|_{L^p(\mathcal{B}_N^*)} |\mathcal{B}_N^*|^{\frac{p-2}{2p}} \leq c_2\varepsilon_1 r$$

Therefore, if ε_1 is taken sufficiently small, such that

$$\varepsilon_1 \leq \left(c_2^2 + c_1 + 24\pi \left(\frac{2K + 4(N-3)}{\beta} + 1 + c_1\right)\right)^{-1} \cdot \frac{\pi}{2}$$

and

$$\varepsilon_1 \leq c_2^{-1} c_3,$$

where c_3 is as in Lemma 3.1, and since

$$\partial\Delta \subset u(\Omega) \subset \mathcal{B}_{N-1} \setminus \mathcal{B}_2^* \implies \sup_{\partial\Delta} g \leq r \leq \sqrt{r},$$

(see (23)) we can apply Lemma 3.1 with M and r replaced by Δ and $\sqrt{r}$ respectively. This application then gives

$$\sup_{\mathcal{B}_2^*} g \leq \sup_{\Delta} g \leq \frac{5}{4}\sqrt{r},$$

which finishes the proof of *(iii)* and of the lemma. □

The next theorem shows that given an embedded geodesic ball $\mathcal{B}_R(x_0)$ in a simply-connected surface, if $\mathcal{B}_R(x_0)$ has bounded L^2 norm of $|A|$ and $\mathcal{B}_R^*(x_0)$ has sufficiently small L^p norm of H then this geodesic ball is graphical away from its boundary.

Theorem 4.4. *Given $K > 0$ and $p > 2$, there exist $\varepsilon = \varepsilon(K, p)$ and $\gamma = \gamma(K)$, such that the following holds. Let M be a simply-connected surface embedded in $\mathbb{R}^3$ and let $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$ be such that*

$$\int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2 \le K r^4 \quad \text{and} \quad |\mathcal{B}_R^*|^{\frac{p-2}{p}} \left(\int_{\mathcal{B}_R^*} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \le \varepsilon r^2$$

for some $r \in [0, 1/4]$. Then, after a rotation

$$\sup_{\mathcal{B}_{\gamma R}^*} g \le \frac{5}{4} r.$$

Proof. Without loss of generality, let us assume that $x_0 = 0$. Note also that by rescaling it suffices to prove the theorem for $R = 1$, i.e. we assume that $\mathcal{B}_1 := \mathcal{B}_1(0) \subset M \setminus \partial M$,

$$\int_{\mathcal{B}_1} |A|^2 \, d\mathcal{H}^2 \le K r^4 \quad \text{and} \quad |\mathcal{B}_1^*|^{\frac{p-2}{2p}} \left(\int_{\mathcal{B}_1^*} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \le \varepsilon r^2.$$

We will show that this theorem is a consequence of Lemma 4.2. In order to do this we begin by proving the following claim.

Claim 4.5. *Given $\varepsilon_1 > 0$, there exists $s \in [20^{-n_0}, 20^{-1}]$, such that*

$$\int_{\mathcal{B}_{20s} \setminus \mathcal{B}_s} |A|^2 \, d\mathcal{H}^2 \le c_1 (\varepsilon_1 r^2)^2,$$

where $n_0 = \left[\frac{K}{c_1 \varepsilon_1^2} \right] + 1$ and c_1 is as in Theorem 1.3.

Proof of Claim 4.5. Note that $\mathcal{B}_{20s} \setminus \mathcal{B}_s \subset \mathcal{B}_1$, $\forall s \le 20^{-1}$ and writing

$$\mathcal{B}_1 = \bigcup_{j=1}^{n_0} (\mathcal{B}_{20^{-(j-1)}} \setminus \mathcal{B}_{20^{-j}}) \bigcup \mathcal{B}_{20^{-n_0}}$$

we have that

$$\sum_{j=1}^{n_0} \int_{\mathcal{B}_{20^{-(j-1)}} \setminus \mathcal{B}_{20^{-j}}} |A|^2 \, d\mathcal{H}^2 \le \int_{\mathcal{B}_1} |A|^2 \, d\mathcal{H}^2 = K r^4$$

which implies that for some $j_0 \in \{1, \dots n_0\}$, we have that

$$\int_{\mathcal{B}_{20^{-(j_0-1)}} \setminus \mathcal{B}_{20^{-j_0}}} |A|^2 \, d\mathcal{H}^2 \le \frac{K r^4}{n_0} \le c_1 (\varepsilon_1 r^2)^2.$$

Hence the claim is true with $s = 20^{-j_0}$, with j_0 being as above. $\square$

Let $\varepsilon_1 = \varepsilon_1(K)$ be as in Lemma 4.2 with $N = 20$ and let $s \in [20^{-n_0}, 20^{-1}]$ be as in the previous claim, i.e. so that

$$\int_{\mathcal{B}_{20s}\setminus\mathcal{B}_s} |A|^2 \, d\mathcal{H}^2 \le c_1(\varepsilon_1 r^2)^2$$

where c_1 is as in Theorem 1.3 and $n_0 = \left[\frac{K}{c_1\varepsilon_1^2}\right] + 1$.

Let $\widetilde{M} = s^{-1}M$ be the rescaling of M by s^{-1} and $\widetilde{A}, \widetilde{H}$ the corresponding second fundamental form and mean curvature and let $\widetilde{\mathcal{B}}$ denote the geodesic balls of $\widetilde{M}$. Then we have

$$\int_{\widetilde{\mathcal{B}}_{20}} |\widetilde{A}|^2 \, d\mathcal{H}^2 = \int_{\mathcal{B}_{20s}} |A|^2 \, d\mathcal{H}^2 \le \int_{\mathcal{B}_1} |A|^2 \, d\mathcal{H}^2 \le Kr^4 \text{ and}$$

$$\int_{\widetilde{\mathcal{B}}_{20}\setminus\widetilde{\mathcal{B}}_1} |\widetilde{A}|^2 \, d\mathcal{H}^2 = \int_{\mathcal{B}_{20s}\setminus\mathcal{B}_s} |A|^2 \, d\mathcal{H}^2 \le c_1(\varepsilon_1 r^2)^2.$$

Furthermore

$$\begin{aligned} |\widetilde{\mathcal{B}}^*_{20}|^{\frac{p-2}{2p}} \left(\int_{\widetilde{\mathcal{B}}^*_{20}} |\widetilde{H}|^p \, d\mathcal{H}^2\right)^{\frac{1}{p}} &= |\mathcal{B}^*_{20s}|^{\frac{p-2}{2p}} \left(\int_{\mathcal{B}^*_{20s}} |H|^p \, d\mathcal{H}^2\right)^{\frac{1}{p}} \\ &\le |\mathcal{B}^*_1|^{\frac{p-2}{2p}} \left(\int_{\mathcal{B}^*_1} |H|^p \, d\mathcal{H}^2\right)^{\frac{1}{p}} \le \varepsilon r^2. \end{aligned}$$

Let $\varepsilon = c_2\varepsilon_1 \left(16C^2\right)^{\frac{2-p}{2p}}$, where c_2 is as in Theorem 1.3, (and Lemma 4.2) and where C is the isoperimetric constant as in (1). Then

$$(16C^2|\widetilde{\mathcal{B}}^*_{20}|)^{\frac{p-2}{2p}} \left(\int_{\widetilde{\mathcal{B}}^*_{20}} |\widetilde{H}|^p \, d\mathcal{H}^2\right)^{\frac{1}{p}} \le c_2\varepsilon_1 r^2$$

and by applying Lemma 4.2 to $\widetilde{\mathcal{B}}_{20} \subset \widetilde{M}$, with $N = 20$ and with r replaced by r^2 we obtain, after possibly a rotation, the following estimate:

$$\sup_{\widetilde{\mathcal{B}}^*_2} g \le \frac{5}{4} r.$$

Since the quantity g is scale invariant, we have that $\sup_{\mathcal{B}^*_{2s}} g \le \frac{5}{4}r$. Let $\gamma = 2 \cdot 20^{-n_0}$, since $s \ge 20^{-n_0}$ this gives that

$$\sup_{\mathcal{B}^*_\gamma} g \le \frac{5}{4} r.$$

This finishes the proof of the theorem. $\square$

We finally show that we can derive Theorem 1.1 in the introduction by the above Theorem 4.4. Theorem 1.1 states that if $\mathcal{B}_R$ is an embedded disk with bounded L^2 norm of $|A|$, then $\mathcal{B}_R$ is graphical away from its boundary, provided that the L^p norm of H is sufficiently small. For convenience we recall the statement of the theorem.

Theorem 1.1. *Given $K > 0$ and $p > 2$, there exists $\varepsilon = \varepsilon(K, p)$ and $\gamma = \gamma(K)$, such that the following holds. Let $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$ be an embedded disk such that*

$$\int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2 \leq K r^4 \quad \textit{and} \quad R^{\frac{p-2}{p}} \left(\int_{\mathcal{B}_R} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq \varepsilon r^2$$

for some $r \in [0, 1/4]$. Then, after a rotation,

$$\sup_{\mathcal{B}_{\gamma R}} g \leq \frac{5}{4} r.$$

Proof. Since $\mathcal{B}_R$ is a disk, we have that $\mathcal{B}_R = \mathcal{B}_R^*$ and furthermore by Lemma 2.3 we have that

$$|\mathcal{B}_R| \leq \left(\pi + \frac{K r^4}{2} \right) R^2.$$

Therefore

$$\begin{aligned} |\mathcal{B}_R^*|^{\frac{p-2}{2p}} \left(\int_{\mathcal{B}_R^*} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} &\leq \left(\pi + \frac{K r^4}{2} \right)^{\frac{p-2}{2p}} R^{\frac{p-2}{p}} \left(\int_{\mathcal{B}_R} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \\ &\leq \left(\pi + \frac{K}{2} \right)^{\frac{p-2}{2p}} \varepsilon r^2 \end{aligned}$$

and hence we can directly apply Theorem 4.4. $\square$

5 Appendix

For the sake of completeness, in this appendix we prove two results in differential geometry that are used throughout the paper. In Remark 5.3 we also discuss the $C^{1,\alpha}$ regularity.

Let

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 | a^2 < x_1^2 + x_2^2 < b^2\}$$

for certain $b > a > 0$ and let $\mathcal{A}$ denote the graph above Ω of a smooth function u. That is $u \in C^\infty(\Omega)$ and $\operatorname{graph} u = \mathcal{A}$.

Lemma 5.1. *Assume that*

$$|Du| \leq r \leq 1 \text{ and } \int_{\mathcal{A}} |A|^2 \, d\mathcal{H}^2 \leq \varepsilon$$

Then there exists $\rho \in (a, b)$ for which

$$\int_{u(S_\rho)} k \, ds \leq 2\pi \left(1 + r\sqrt{2} + \left(\frac{2\varepsilon b}{b-a} \right)^{\frac{1}{2}} \right),$$

where k is the curvature of the curve $u(S_\rho)$ and $S_\rho = \{(x_1, x_2) : x_1^2 + x_2^2 = \rho^2\}$.

Proof. Recall that in graphical coordinates

$$A_{ij}(x, u(x)) = \left(\frac{\partial^2}{\partial x_i \partial x_j} (x_1, x_2, u(x_1, x_2)) \right)^{\perp} = (0, 0, D_{ij}u) \cdot \nu = \frac{D_{ij}u}{\sqrt{1 + |Du|^2}},$$

where A_{ij}, $i = 1, 2$, are the coefficients of the second fundamental form, and also that

$$|A|^2 = A_{ij} A_{kl} g^{ik} g^{il},$$

where g is the induced metric. We have that

$$|D^2 u|^2 = \sum_{i,j=1}^{2} |D_{ij}u|^2 \leq |A|^2 (1 + |Du|^2)^3$$

(see for example [5]). On Ω we are assuming that

$$|Du(x)| \leq r \, , \; \forall x \in \Omega$$

and this, together with the area formula, gives

$$\begin{aligned} \int_{\Omega} |D^2 u(x)|^2 \, dx &= \int_{\Omega} \frac{|D^2 u|^2}{(1 + |Du|^2)^3} (1 + |Du|^2)^3 dx \\ &\leq (1 + r^2)^{\frac{5}{2}} \int_{\Omega} \frac{|D^2 u|^2}{(1 + |Du|^2)^3} \sqrt{1 + |Du|^2} dx \\ &= (1 + r^2)^{\frac{5}{2}} \int_{\mathcal{A}} |A|^2 \, d\mathcal{H}^2 \leq 2(1 + r)\varepsilon. \end{aligned}$$

By the coarea formula we can pick $\rho \in (a, b)$, so that

$$\int_{S_\rho} |D^2 u|^2 \, dx \leq \frac{2(1 + r)\varepsilon}{b - a}.$$

Let $\Gamma = u(S_\rho)$. Γ is a closed curve in $\mathcal{A}$ and we want to compute

$$\int_\Gamma k\, ds.$$

Let $\gamma : [0,1] \to S_\rho$ be the following parametrization of S_ρ:

$$\gamma(t) = (\rho \cos 2\pi t, \rho \sin 2\pi t)$$

and consider the parametrization of Γ given by

$$f(t) = (\gamma(t), u(\gamma(t))), \; t \in [0,1].$$

Recall that

$$\int_\Gamma k\, ds = \int_0^1 k(f(t))|f'(t)|\, dt \le \int_0^1 \frac{|f''|}{|f'|} dt,$$

since

$$k = \frac{|f' \times f''|}{|f'|^3} \implies k \le \frac{|f''|}{|f'|^2}.$$

Furthermore

$$|f'|^2 = |\gamma'(t)|^2 + \left|\frac{d}{dt} u(\gamma(t))\right|^2 \implies$$

$$(2\pi\rho)^2 = |\gamma'(t)|^2 \le |\gamma'(t)|^2 + |Du|^2 |\gamma'(t)|^2 = |f'|^2$$

and

$$\begin{aligned}
|f''| = \left|\left(\gamma''(t), \frac{d^2}{dt^2} u(\gamma(t))\right)\right| &\le |\gamma''(t)| + \left|\frac{d^2}{dt^2} u(\gamma(t))\right| \\
&\le \rho(2\pi)^2 + \sqrt{2}|D^2u||\gamma'(t)|^2 + \sqrt{2}|Du||\gamma''(t)| \\
&\le \rho(2\pi)^2 + \sqrt{2}|D^2u|(2\pi\rho)^2 + \sqrt{2} r\rho(2\pi)^2,
\end{aligned}$$

where we have used the computation:

$$\begin{aligned}
\left|\frac{d^2}{dt^2} u(\gamma)\right| = \left|\frac{d}{dt}(Du \cdot \gamma')\right| &\le |\gamma'|^2 \left(\sum_{i,j=1}^2 |D_{ij}u|\right) + |Du| \left(|\gamma_1''| + |\gamma_2''|\right) \\
&\le |\gamma'|^2 \sqrt{2} \left(\sum_{i,j=1}^2 |D_{ij}u|^2\right)^{\frac{1}{2}} + |Du|\sqrt{2}|\gamma''| \\
&= |\gamma'|^2 \sqrt{2} |D^2u| + |Du|\sqrt{2}|\gamma''|.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_\Gamma k\, ds &\le \int_0^1 \frac{\rho(2\pi)^2 + \sqrt{2}|D^2u|(2\pi\rho)^2 + r\rho\sqrt{2}(2\pi)^2}{2\pi\rho} dt \\
&= 2\pi(1 + \sqrt{2} r) + 2\sqrt{2}\pi\rho \int_0^1 |D^2u(\gamma(t))| dt.
\end{aligned}$$

Using again the area formula we have

$$\int_0^1 |D^2u(\gamma(t))|dt = \int_0^1 \frac{|D^2u(\gamma(t))|}{|\gamma'(t)|}|\gamma'(t)|dt = \frac{1}{2\pi\rho}\int_0^1 |D^2u(\gamma(t))||\gamma'(t)|dt$$
$$= \frac{1}{2\pi\rho}\int_{S_\rho} |D^2u(x)|dx \le \frac{1}{2\pi\rho}\left(\int_{S_\rho} |D^2u(x)|^2 dx\right)^{\frac{1}{2}} |S_\rho|^{\frac{1}{2}}$$
$$\le \frac{1}{2\pi\rho}\left(\frac{4\pi\rho(1+r)\varepsilon}{b-a}\right)^{\frac{1}{2}} = \left(\frac{(1+r)\varepsilon}{\pi\rho(b-a)}\right)^{\frac{1}{2}}.$$

Hence

$$\int_\Gamma k\, ds \le 2\pi(1+r\sqrt{2}) + 2\sqrt{2}\pi\rho\left(\frac{(1+r)\varepsilon}{\pi\rho(b-a)}\right)^{\frac{1}{2}} = 2\pi\left(1+r\sqrt{2}+\left(\frac{2(1+r)\varepsilon\rho}{\pi(b-a)}\right)^{\frac{1}{2}}\right)$$
$$\le 2\pi\left(1+r\sqrt{2}+\left(\frac{2\varepsilon b}{b-a}\right)^{\frac{1}{2}}\right).$$

□

Lemma 5.2. *Let $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$ and assume that*

$$g(x) = \frac{1}{\sqrt{2}}|\nu(x) - e_3| \le r\,,\ \forall x \in \mathcal{B}_R,$$

for some $r \in \left[0, \frac{1}{\sqrt{2}}\right]$. Then $\mathcal{B}_R$ is locally graphical over the plane $\{x_3 = 0\}$ with gradient bounded by $3r$. Moreover, $\mathcal{B}_R$ contains a graph of a function u over the disk in the plane $\{x_3 = 0\}$ centered at $\Pi(x_0)$ and of radius $\rho = \frac{R}{\sqrt{1+(3r)^2}}$; where Π denotes the projection on the plane $\{x_3 = 0\}$.

Proof. Since $g(x) \le 1$ for all $x \in \mathcal{B}_R$, we have that $\mathcal{B}_R$ is locally a graph over the plane $\{x_3 = 0\}$ and at each point $x = (x_1, x_2, u(x_1, x_2)) \in \mathcal{B}_R$, we have

$$\nu(x) = \left(-\frac{D_1u}{\sqrt{1+|Du|^2}}, -\frac{D_2u}{\sqrt{1+|Du|^2}}, \frac{1}{\sqrt{1+|Du|^2}}\right) \tag{29}$$

where ν is the upward pointing unit normal. We estimate now $|Du|^2 = |D_1u|^2 +$

$|D_2u|^2$ using the estimate for g as follows: Note first that

$$g(x) = \frac{1}{\sqrt{2}}|\nu(x) - e_3| = \frac{1}{\sqrt{2}}\left(\frac{|D_1u|^2}{1+|Du|^2} + \frac{|D_2u|^2}{1+|Du|^2} + \left(1 - \frac{1}{\sqrt{1+|Du|^2}}\right)^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{|Du|^2}{1+|Du|^2} + \frac{1}{1+|Du|^2} + 1 - 2\frac{1}{\sqrt{1+|Du|^2}}\right)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}\left(2 - 2\frac{1}{\sqrt{1+|Du|^2}}\right)^{\frac{1}{2}} = \left(1 - \frac{1}{\sqrt{1+|Du|^2}}\right)^{\frac{1}{2}}.$$

Hence, since $g(x) \leq r$, we get

$$g(x) = \left(1 - \frac{1}{\sqrt{1+|Du|^2}}\right)^{\frac{1}{2}} \leq r \implies 1 - \frac{1}{\sqrt{1+|Du|^2}} \leq r^2 \implies$$

$$\sqrt{1+|Du|^2} \leq \frac{1}{1-r^2} \leq 1 + 2r^2,$$

with the last inequality being true since $r \leq \frac{1}{\sqrt{2}} \implies r^2 - 2r^4 \geq 0$. Squaring both sides we obtain

$$1 + |Du|^2 \leq 1 + 4r^4 + 4r^2 \implies |Du|^2 \leq 9r^2 \implies |Du| \leq 3r.$$

This finishes the proof of the first part of the lemma.

By the previous discussion, $\mathcal{B}_R$ is a graph of a function u around the point x_0. Let ρ be such that u is defined on the disk centered at $\Pi(x_0)$ of radius ρ in the plane $\{x_3 = 0\}$, $D_\rho(\Pi(x_0))$. Without loss of generality, let $x_0 = 0$. We will prove a lower estimate for the radius ρ of the disk where the function u is defined. To do this, let ρ be the maximum such radius. Then there exists a point $(x_1, x_2) \in \partial D_\rho(0)$, for which $(x_1, x_2, u(x_1, x_2)) \in \partial\mathcal{B}_R$, else u maps $\partial D_\rho(0)$ in the interior of $\mathcal{B}_R$ and since $\mathcal{B}_R$ is locally a graph over the plane $\{x_3 = 0\}$ we could increase ρ. Let $\gamma(t)$ be the path in $\mathcal{B}_R$ defined by

$$\gamma : [0,1] \to \mathcal{B}_R, \quad \gamma(t) = (tx_1, tx_2, u(t(x_1, x_2))).$$

The path γ joins 0, that is the center of $\mathcal{B}_R$, with $x \in \partial\mathcal{B}_R$, therefore it must have length at least R, from which we get

$$R \leq \text{Length}(\gamma) = \int_0^1 |\dot{\gamma}| dt \leq \int_0^1 \rho\sqrt{1+|Du|^2} dt \leq \rho\sqrt{1+(3r)^2}$$

which implies that

$$\rho \geq \frac{R}{\sqrt{1+(3r)^2}}$$

and this finishes the proof of the lemma. $\square$

Remark 5.3. *Standard PDE theory implies that under the hypotheses of* Lemma 5.2 *and if in addition $r \leq \frac{1}{\sqrt{3}}$ and $H \in L^p(\mathcal{B}_R)$, $p > 2$, we obtain $C^{1,\alpha}$ estimates in $\mathcal{B}_{\frac{R}{2}}$; namely, there exists a constant $C = C\left(r, \int_{\mathcal{B}_R} |H|^p d\mathcal{H}^2\right)$ such that*

$$\frac{|\nu(x) - \nu(y)|}{|x-y|^\alpha} \leq R^{-\alpha} C, \quad \forall x, y \in \mathcal{B}_{R/2}$$

To see why the above remark is true, we can assume without loss of generality that $x_0 = 0$. Note that by Lemma 5.2, $\mathcal{B}_R$ contains a graph of a function u over $\Omega := \{(x_1, x_2) : x_1^2 + x_2^2 \leq \rho^2\}$ with $|Du| \leq 3r$ and with

$$\rho = \frac{R}{\sqrt{1+(3r)^2}} \geq \frac{R}{\sqrt{10}} \geq \frac{R}{2}.$$

Thus $\mathcal{B}_{\frac{R}{2}} \subset \operatorname{graph} u$. Furthermore u satisfies the equation

$$\sum_{i=1}^{2} D_i \left(\frac{D_i u(x)}{\sqrt{1+|Du(x)|^2}} \right) = H(x, u(x)).$$

By differentiating the above equation, we obtain that $w = D_k u$, for $k = 1, 2$, is a solution to the equation

$$\sum_{i,j=1}^{2} D_i(a^{ij} D_j w) = D_k H$$

(cf. [7, pages 319-320]) with

$$a^{ij} = \frac{\delta_{ij}}{\sqrt{1+|Du|^2}} - \frac{D_i u D_j u}{\sqrt{1+|Du|^2}}.$$

Note that since $|Du|$ is bounded we have that $H \in L^p(\mathcal{B}_R) \implies H \in L^p(\Omega)$. We can then apply Theorem 8.22 in [7] to obtain that

$$\sup_{x,y \in \Omega} \frac{|Du(x) - Du(y)|}{|x-y|^\alpha} \leq \rho^{-\alpha} C$$

for some $\alpha \in (0,1)$ and with C and α depending on $\sup_\Omega |Du|$, $\|H\|_{L^p(\Omega)}$ and p, namely on r, $\|H\|_{L^p(\mathcal{B}_R)}$ and p, and $\rho \geq \frac{R}{2}$. Using the formula for ν as in (29), we get the required estimate.

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